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Cauchy stress in mass distributions

The thermodynamic definition of pressure $P = \partial U/\partial V$ is one form of the principle that in a given state, the mass in V and potential are proportional. Subject of this communication is the significance of this principle for the understanding of Cauchy stress.

The stress theory as it is used today, was developed by Euler in 1776 and Cauchy in 1823. The following is a slightly edited quote of Truesdell [1].

p.164: Let **f** be pairwise equilibrated; let $-S$ denote the contact having the same underlying set as S but opposite orientation; then

$$\mathbf{t}_{-S} = -\mathbf{t}_S \tag{1}$$

p.170: Cauchy assumed that the tractions **t** on all like-oriented contacts with a common plane at **x** are the same at **x**, i.e. $\mathbf{t}_S$ at **x** is assumed to depend on S only through the normal **n** of S at **x**: $\mathbf{t}_S = \mathbf{t}(\mathbf{x}, \mathbf{n})$. This statement is called the Cauchy postulate. S is oriented so that its normal **n** points out of c(B) if S is a part of ∂c(B). Thus $\mathbf{t}(\mathbf{x}, -\mathbf{n})$ is the traction at **x** on all surfaces **S** tangent to ∂c(B) and forming parts of the boundaries of bodies in the exterior c(B^e) of c(B). In this sense $\mathbf{t}(\mathbf{x}, \mathbf{n})$ is the traction exerted upon B at **x** by the contiguous bodies outside it. As a trivial corollary of (1) follows Cauchy´s fundamental lemma: $\mathbf{t}(\mathbf{x}, -\mathbf{n}) = -\mathbf{t}(\mathbf{x}, \mathbf{n})$.

p.176: $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. At a given place $\mathbf{x}_0$ the planes P_1 and P_2 normal to $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, are distinct. We set $\mathbf{v}_3 = -(\mathbf{v}_1 + \mathbf{v}_2)$ and consider the wedge **A** that is bounded by these two planes and the plane P_3 normal to $\mathbf{v}_3$ at the place $\mathbf{x}_0 + \varepsilon\mathbf{v}_3$. We suppose ε small enough that **A** be the shape of some part of B, and we denote by $\partial_i\mathbf{A}$ the portion of the plane P_i that makes a part of the boundary of **A**. We let ε approach 0. If we write A_i for the area of $\partial_i\mathbf{A}$, we see that

$$\begin{aligned} A_1 &= \frac{|\mathbf{v}_1|}{|\mathbf{v}_3|}A_3, \qquad A_2 = \frac{|\mathbf{v}_2|}{|\mathbf{v}_3|}A_3, \\ A_3 &= O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0, \\ V(\mathbf{A}) &= \frac{\varepsilon|\mathbf{v}_3|A_3}{2}. \end{aligned} \tag{2}$$

$$\text{If } \mathbf{c} = \frac{|\mathbf{v}_3|}{A_3}\int_{\partial\mathbf{A}} \mathbf{t}(\mathbf{x}, \mathbf{n})\, dA, \tag{3}$$

from (2) and the assumption that $\mathbf{t}(\cdot, \mathbf{n})$ is continuous we see that

$$\mathbf{c} = \sum_{i=1}^{3} \frac{|\mathbf{v}_i|}{A_i}\int_{\partial_i\mathbf{A}} \mathbf{t}\left(\mathbf{x}, \frac{\mathbf{v}_i}{|\mathbf{v}_i|}\right) dA + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{4}$$

Since **t** is a homogeneous function of its second argument and a continuous function of its first argument,

$$\mathbf{c} \to \sum_{k=1}^{3} \mathbf{t}(\mathbf{x}_0, \mathbf{v}_k) \quad \text{as} \quad \varepsilon \to 0. \tag{5}$$

On the other hand, we see that $\mathbf{c} \to \mathbf{0}$ as $\varepsilon \to 0$. Therefore, since the sum in (5) is independent of ε, it must vanish:

$$\sum_{k=1}^{3} \mathbf{t}(\mathbf{x}_0, \mathbf{v}_k) = \mathbf{0}. \text{ [End of quote]} \tag{6}$$

The key argument in the above text is: "*since* the sum in (5) is independent of ε". It is an *a priori* condition; behind this is the assumption that Newton's 3rd law (1) is the proper equilibrium condition for the problem, and the Newtonian understanding of pressure, $P = |\mathbf{f}|/A$ which is believed to be universally scale-independent. Pressure is a state function, and a pressure increase requires that work is done on a system of mass distributed in V; it is a change of state in the sense of the First Law. Pressure is defined as energy density,

$$P = \partial U/\partial V. \tag{7}$$

The question is then: how are $P = |\mathbf{f}|/A$ and $P = \partial U/\partial V$ mathematically related?

The thermodynamic definition of P is scale-independent, and an explicit statement of the proportionality of mass (measured in V the radius of which is $r = |\mathbf{r}|$) and potential U in a given state (Kellogg [2:80]). The thermodynamic equilibrium condition is

$$P_{\text{surr}} - P_{\text{syst}} = 0 \tag{8}$$

in scalar form. If both terms are thought to be caused by forces **f** [Newton] acting from either side on the surface of the system V the equilibrium condition is

$$\mathbf{f}_{\text{surr}} - \mathbf{f}_{\text{syst}} = 0; \tag{9}$$

for isotropic conditions (subsequently implied), both **f** are radial force fields. Since the system contains mass, and since it interacts with the surrounding through exchange of work, it acts as a source of forces; i.e. its source density $\varphi \neq 0$ in some statically loaded state. φ is always proportional to the mass in the system (Kellogg [2:45]); an existence theorem requires that if there is some function f of a point Q such that

$$\int f(Q)\, dV = \varphi\,, \tag{10}$$

both LHS and RHS must vanish with the maximum chord of V (Kellogg [2:147]). As with all of thermodynamics (Born [3]), the approach to stress must thus be based on a Poisson equation (Kellogg [2:156]). The equilibrium condition (8) thus can take the form

$$\varphi_{\text{surr}} - \varphi_{\text{syst}} = 0\,. \tag{11}$$

It is therefore of interest how the volume functions relate to the surface functions if the domain of interest V is changed in scale. In

$$\int \mathbf{f} \cdot \mathbf{n}\; dA \;=\; \int \nabla \cdot \mathbf{f}\; dV \;=\; \varphi\,, \tag{12}$$

f may be either one of the LHS terms in (9). If mass is continuously distributed, $\varphi \propto V$, and $\nabla \cdot \mathbf{f}$ is a constant that is characteristic of the energetic state in which the system is. Hence in (12), LHS $\propto V$. Since $V \propto r^3$, but $A \propto r^2$, for LHS $\propto r^3$ to hold it follows that $|\mathbf{f}| \propto r$, or

$$\frac{|\mathbf{f}|}{|\mathbf{r}|} = \; const\,. \tag{13}$$

Thus as $V \to 0$, $|\mathbf{f}|/A \to \infty$, yet $\Delta U/\Delta V \to const$. Both $\mathbf{f}_{\text{syst}}$ and $\mathbf{f}_{\text{surr}}$ vanish with **r**; the condition in (10) is observed, stating that a system V with zero magnitude cannot do work on its surrounding, and vice versa.

ε (2-5) is an one-dimensional measure of the magnitude of the prism **A** (2), as is r for V in the subsequent discussion. It is to be taken into account that $P = |\mathbf{f}|/A$ is scale-independent if A is a free plane, yet both the surface of the prism **A** in (2) and the surface A in (12) are closed surfaces. The difference between (1) and (9) is that the latter distinguishes system and surrounding whereas the former does not. The thermodynamic system V represents a distributed source in the sense of potential theory (Kellogg [2:150 ff]). ε or r, respectively, is the zero potential distance (Kellogg [2:63]) which may have infinite length, or if it is finite it is set to have unit length by convention, but it cannot be zero or otherwise be let vanish.

References

1 TRUESDELL, C.A.: A first course in rational continuum mechanics. Academic Press, 1991.
2 KELLOGG, O.D.: Foundations of potential theory. Springer Verlag, 1929.
3 BORN, M.: Kritische Betrachtungen zur traditionellen Darstellung der Thermodynamik. Physik. Zeitschr., **22** (1921), 218-224, 249-254, 282-286.

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